

Structure of Constant Rank State Machines, and the five Basic State Machine types

NICO F. BENSCHOP - *Amspade Research* - The Netherlands

IFIP TC10 Workshop on "Logic and Architecture Synthesis"
Paris, 30 May - 1 June 1990

Abstract

Constant Rank (CR) state machines play an important role in the general structure theory of Finite State Machines. A machine is of constant rank if each input and input-sequence maps the state set onto the same number of next states. CR-machines are analysed, using their sequential closure (semigroup), which is shown to be a **simple semigroup**, that is: a semi- direct product $(L \times R) \triangleright G$ of a left- and a right-copy semigroup, and a group. So in general a CR-machine is a composition of: a *branch*-, a *reset*- and a *permutation* machine, which are three of the five basic types of state machines to be derived [1].

1 Introduction: Sequential Closure and Rank

A brief review of [1] is necessary to set up the required concepts. A state machine $M(Q, A)$ with stateset Q and input alphabet A is a function $M : Q \times A \rightarrow Q$, which maps present state and input to next state. It is specified by a state transition table with $|A|$ columns and $|Q|$ rows. Each input $a \in A$ is interpreted as a function $a : Q \rightarrow Q$, mapping stateset Q into itself, called a **state transform**, or in short: a transform.

Sequential composition ab of two transforms a and b is defined by $q(ab) = (qa)b$, for all q in Q . In other words, in state q first apply input a to get state qa , then apply b , which yields state $(qa)b = q(ab) = qab$. Notice the left-to-right notation of this **function composition**, with stateset Q as domain and codomain. Two input sequences over A are defined **equivalent** if they yield the same Q -transform: $a = b$ iff $qa = qb$ for all q in Q .

The **sequential closure** of M , called **semigroup** S , is the (finite) set of Q -transforms generated by all sequences over A , denoted $S = A^+/Q$. Here A^+ denotes the infinite semigroup of non empty strings of length ≥ 1 over alphabet A , under string concatenation.

Closure S of machine M is a finite semigroup (of order $\leq n^n$, if M has n states) since transform composition is **associative**: $a(bc) = (ab)c$ for all $a, b, c \in S$, which is clear from above definition of transform composition. Input strings with the same Q -transform

are defined *equivalent* with respect to machine M , so the transform representation of each element of S is unique. State transform $x : Q \rightarrow Q$ is a function defined on a state set Q , which is both domain and co-domain. To state transform x correspond:

- **range** Qx is the set of function values (next states),
- **partition** Px equivalences states mapping onto the same next state,
- **rank** $r(x) =$ the order $|Qx|$ of its range = the number of partition blocks.

LEMMA 1.1 *Non-increasing Rank property :*

- (a) *Left composition x . does not increase range : $Qxy \subseteq Qy$ (\subseteq : subset of)*
- (b) *Right composition $.y$ does not refine partition: $Pxy \geq Px$ ($>$: coarser than)*
- (c) *Rank does not increase under transform composition: $r(xy) \leq r(x)$ and $r(xy) \leq r(y)$*
- (d) *All elements x with $rank(x) \leq k$ form a subsemigroup which is an ideal Z_k of S .*

PROOF. (a) $Qxy \subseteq Qy$ follows from set inclusion and associativity. $Qx \subseteq Q$, for all x , and right composition with y yields: $(Qx)y = Q(xy) \subseteq Qy$.

(b) $Pxy \geq Px$ follows from associativity and right composition of states i, j that are equivalent under $x : ix = jx$ implies $ixy = jxy$ for all y .

(c) This **monotone rank property** follows directly from (a) and (b), because range ordering (a) implies rank ordering $|Q(xy)| \leq |Qy|$, so $r(xy) \leq r(y)$, and partition ordering (b) implies rank ordering $|P(xy)| \geq |Px|$, so $r(xy) \leq r(x)$.

(d) It follows immediately that if x and y have $rank \leq k$, then so does composition xy . This closure property means that all elements of rank not exceeding k form a subsemigroup Z of S . In fact, composition of any element $z \in Z$ with any element $s \in S$ yields zs with $r(zs) \leq r(z) \leq k$, so that zs is in Z . The same holds for sz . Hence Z is both left- and right ideal, that is an ideal of $S : ZS \subseteq Z$ and $SZ \subseteq Z$ (see def-2 next section). \square

Basically, this paper tries to render results from semigroup structure and their state representation better accessible for state machine decomposition purposes. In fact, the earliest known result in semigroup theory (Suschkewitch, 1928 [2, p207]) is on the structure of the minimal ideal of a semigroup, essentially theorem 3.1.

2 Basic Machines and Simple Semigroups

Machine decomposition is seen as implementing a machine as a network of *smaller* machines. Semigroups, as the sequential closures of state machines, are essential for the **equivalencing** and **ordering** of machines. Two machines are defined to be equivalent if they have isomorphic semigroups. Two machines are ordered $M1 \leq M$ if their closures are ordered $S1 \subseteq S$, meaning that $S1$ is (isomorphic to) a subsemigroup of S . Naturally, a minimal or **basic machine** has a closure with *no proper subsemigroup* [1].

The **five basic state machines**, with semigroups having no proper sub-semigroup (prime cycles C_p and those of order two) are derived in [1]. Their interpretation as the elementary digital functions are: *logic*, *arithmetic* and *memory*. A semigroup S is also a state machine

$M(S, S)$ with itself as inputset and state set. For unique representation by state transforms (distinct columns), one extra state suffices if some columns are equal in the $S \times S$ composition table, see tables U2 and L2. Components C2 and U2 have a single generator '1', the others have two invariant generators $a^2 = a$.

Def 1: a semigroup is of **constant rank** (CR) if it can be represented by transforms of equal rank. A state machine is of constant rank if its closure is a CR-semigroup.

Three basic components are of constant rank, namely *L2*, *R2* and *C2*. They are the smallest cases of the following three types of **constant rank semigroups**:

L : Left-copy semigroup with $ab = a$ for all $a, b \in S$ (n -branch, $n+1$ states)

R : Right-copy semigroup with $ab = b$ for all $a, b \in S$ (n -reset, n states)

G : Group (Permutation machine: permutes n states, $|G| \leq n!$)

All three are special cases of the following general type of semigroup [2, p5]:

Def 2: an **ideal** of a semigroup S is a subset Z with $SZ \subseteq Z$ and $ZS \subseteq Z$.

A semigroup is called **simple** if it has *no proper ideal*.

An ideal is like a multiplicative 'zero' ($a.0=0$ for all a) or 'sink'. Notice that U2 (monotone counter with a final state) and H2 (hierarchy of two ordered invariants, see next section) are not simple semigroups, nor are they of constant rank. In general they model the *monotone sequential* aspects and *combinational logic* aspects of state machines respectively.

C2 1 0	U2 1 0	H2 1 0	L2 1 0	R2 1 0	
---+---	---+---	---+---	---+---	---+---	Closure
1 0 1	1 0 0	1 1 0	1 1 1	1 1 0	Tables
0 1 0	0 0 0	0 0 0	0 0 0	0 1 0	
	2 1 0		2 1 0		Fig.1a
$\bar{\quad}$					
/ \		o 1	->o 1		
o-->o	o-->o-->o		/	o o	State-
1 0	2 1 0	o 0	2 o-->o 0	0 1	Diagrams
2-counter	2-counter	AND, OR	2-branch	set/reset	Component
periodic	monotone	isomorph	mux	D-FF	Functions
add(mod 2)	converge	mpy(mod 2)	if-else	assign :=	
<---- iterative a* ---->	<---- invariant : aa=a ----->				Algebraic
	<- LOGIC ->	<-select->	<-store->		Properties
<--- ARITHMETIC : commutative ----->	<- MEMORY non cmt ->				Fig.1b

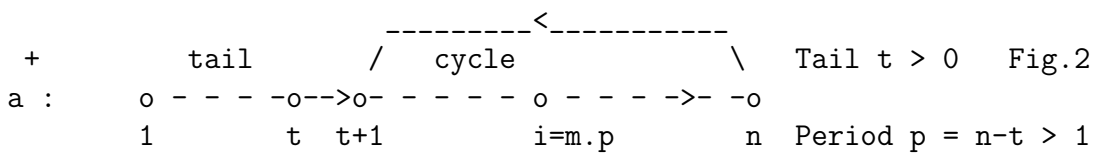
COROLLARY 2.1 A **simple semigroup** is of **constant rank**.

This follows directly from lemma 1d, since otherwise the elements of minimum rank would form a proper ideal. In fact, it will be shown that any simple semigroup is a *semi direct*

product $(L \times R) \mid > G$ of the three basic types of simple semigroups L, R, G . Hence, a general CR-machine is the parallel composition of a *branch* machine, a *reset* machine and a *permutation* machine. In a way, this is a conservation law of sequential logic.

3 Iterations: monotone, periodic, invariant

Iteration in a semigroup S is the repetition a^i of a single element. By virtue of associativity, the result is a unique element in S , independent of bracketing. The closure of a single element a in S is the finite set of its **iterations** $a^+ = \{a^i, i = 1..n\}$ which in general has a tail-cycle structure (‘/Q’ is omitted if no confusion can arise):



Since a^+ is finite, there is a smallest n for which $a^{n+1} = a^{t+1}$ with $tail(a) = t$, ($0 \leq t < n$) and $period(a) = p = n - t$. There is precisely one **invariant** $a^i = (a^i)^2$ where $i = m.p$ is the first and only multiple of p in the cycle, and $a^k = a^{k+p}$ for $k > t$.

An element of semigroup S is called **periodic [monotone]** if its closure has no tail, $t=0$ [no cycle, $p=1$]. Clearly, *invariants* $aa = a$ are the only elements which have both properties. Elements which have a tail and a cycle are called *aperiodic*.

Def 3: a pair e, z of **commuting invariants**: $ez = ze$, $e^2 = e$, $z^2 = z$, is said to be **ordered** $e \geq z$ when $ez = ze = z$, thus e is identity for z . This is an ordering relation, since it is easily seen to be reflexive, anti-symmetric and transitive [2, p23].

3.1 Ordered Invariants: H for combinational logic

It will be shown that any *simple semigroup* S , being of constant rank, contains only *periodic* elements. Moreover, its invariants are not ordered but are all *equivalent* in some sense. So basic components of type U2 (monotone iteration) and H2 (hierarchy of ordered invariants, or combinational logic) do not occur. In fact it turns out that S is a disjoint union of isomorphic groups (G) , with identities forming a direct product of a left-copy (L) and a right-copy (R) semigroup.

LEMMA 3.1 *Ordered Invariants :*

The ordering of commuting invariants $z \leq e$ is their range ordering: $Qz \subseteq Qe$,

hence: - distinct commuting invariants have distinct ranges, and

- ordered invariants $z < e$ have ordered ranks $r(z) < r(e)$.

PROOF. Let invariants z and e be ordered $z \leq e$, then e is identity for z : $ez = ze = z$, so their ranges are ordered because $Qz = Q(ze) = (Qz)e \subseteq Qe$. Notice that $ze = z$ suffices:

e is right identity for z . Conversely, for commuting invariants: $Qz \subseteq Qe$ implies $z \leq e$. This follows from the *state transform structure of an invariant* e : $qee = qe$ means that each state q maps to state qe which is fixed under e . In other words, no state chains of length > 1 occur in the state transition diagram of e .

Range Qe is the set of **fixed states of e** . If $Qz \subseteq Qe$ then z maps each state q into a fixed state of e so $(qz)e = qz$ for all q , hence $ze = z$. Since by assumption e and z commute, we have $ez = ze = z$, meaning $z \leq e$. In case $Qe = Qz$ for commuting invariants e, z then $e \leq z$ and $z \leq e$, hence $e = z$. So commuting invariants with the same range are equal. \square

COROLLARY 3.1 *A simple semigroup S has no ordered invariants, and no pair of invariants commutes (anti-commutative).*

PROOF. Ordered invariants have different ranks according to the previous lemma. Let k be the lowest rank of an ordered pair of invariants. Then, with lemma 1d, S has a proper ideal consisting of all elements with rank $\leq k$ which contradicts S being simple.

If invariants e, f commute: $ef = fe$, then their composition $d = ef$ is also invariant: $d^2 = d$ since $ef.ef = ef.fe = e.ff.e = e.fe = e.ef = ee.f = ef$. Moreover d is ordered under e , since $ed = eef = efe = de = d$ so $d \leq e$, and similarly $d \leq f$. In fact, d is the **greatest lower bound** or **meet** of e and f , which is easily verified [2, p24]. So a commuting pair of invariants is either ordered, or their composition is ordered under both, contradicting simple S . Hence no pair of invariants commutes. \square

So a semigroup of *commuting invariants* is partially ordered set where each pair has a meet (set intersection), called a lower semilattice, with a global zero. For n states, there are at most 2^n commuting invariants (Boolean lattice).

3.2 Equivalent Invariants: L, R for branch- and reset- memory

Consider now the invariants of a simple semigroup S . They do not commute (cor 3.1). Invariants that do not commute may be equivalent in the following sense:

Def 4: Equivalent Invariants

- Invariants a, b that form a left- [right-] copy semigroup $L2$ [$R2$], are called **left- [right] equivalent**, denoted aLb [aRb].
- Invariants a, b are **equivalent**, denoted $a \sim b$, if they are left- or right equivalent: either *directly* so they form $L2$ or $R2$, or *indirectly*: alternating L - and R - equivalent via other invariants.

LEMMA 3.2 *Consider invariants a, b in any semigroup S , represented over stateset Q :*

(a) *Equivalent invariants have equal rank: $a \sim b \rightarrow |Qa| = |Qb|$*

(but equal rank is not sufficient for equivalence: see b)

(b) *Let $(ab)^k = ab$ and $(ba)^k = ba$, with invariants $(ab)^{k-1} = ab^0$ and $(ba)^{k-1} = ba^0$, with max-subgroups $G_{ab^0} = \{x^i = ab^0 \text{ for some } i > 0\}$ resp. G_{ba^0} , then:*

for $k=2$: $\{a, b, ab, ba\}$ are 2 or 4 invariants of equal rank forming $L2, R2$ or $L2 \times R2$

(c) *For $k > 2$: $Lm \times Rn$ holds for max-subgroups $\{G_a, G_b, G_{ab^0}, G_{ba^0}\}$ under set product.*

PROOF. (a) There are three cases of equivalence for invariants a, b : left-, right- and indirect equivalence. In the first two cases of 'direct' equivalence, rank-lemma 1.1 yields:

aLb implies $r(a) = r(ab) \leq r(b)$ and $r(b) = r(ba) \leq r(a)$, so that $r(a) = r(b)$;

aRb implies $r(a) = r(ba) \leq r(b)$ and $r(b) = r(ab) \leq r(a)$, so that $r(a) = r(b)$.

Hence left- or right equivalent invariants have the same rank. Transitivity holds in both cases. For instance let $aLx(ax = a, xa = x)$ and $xLb(bx = b, xb = x)$ then aLb , since $ab = ax.b = a.xb = ax = a$, and similarly $ba = b$. Also right equivalence is transitive.

Now if aLc and cRb with c different from a and b , then a, b are not directly left- or right equivalent, but they are still called (indirectly) equivalent, denoted $aLRb$. Here LR is an equivalence relation since it is easily seen to be reflexive, symmetric and transitive. It follows that if a and b are indirectly equivalent via other invariants, then by transitivity they have all the same rank.

(b) There are several cases: direct and indirect equivalence.

For $k=2$, in the *direct* equivalent case aLb and aRb the elements ab and ba are not different from a and b , forming $L2$ and $R2$ respectively. For *indirect* equivalence of invariants a and b , and in case $k=2$ the only other intermediate elements are invariants ab and ba , with $aba = a$ and $bab = b$, seen as follows. Invariants a, b must have equal rank: $|Qa| = |Qb|$ (lemma 3.2a), so exact equality holds in $(Qa)b \subseteq Qb$, with $Qa.b = Qb$ (*) and similarly $Qb.a = Qa$ (**). Composing both sides of (*) on the right by $.a$ and applying (**) yields $Qa.ba = Qba = Qa$. So sequence $.ba$ permutes $Qa \rightarrow Qa$. Since ba is invariant, this is the identity permutation, hence $(qa)ba = qa$ for all q , meaning $aba = a$. Similarly, invariance of ab implies $bab = b$.

So strings of length > 2 are equivalent to strings of length ≤ 2 , which are just a, b, ab, ba forming a closure of four invariants, with the next equivalences (using $aba = a, bab = b$):

– $aRab$ since $a.ab = aa.b = ab$ and $ab.a = a$, $abLb$ since $ab.b = a.bb = ab$ and $b.ab = b$,

– $bRba$ since $b.ba = bb.a = ba$ and $ba.b = b$, $baLa$ since $ba.a = b.aa = ba$ and $a.ba = a$.

These relations are depicted in a rectangular form in figure 3. The four elements $\{a, b, ab, ba\}$ form an invariant semigroup with direct product structure $L2 \times R2$.

The direct product $L2 \times R2$ (for $k=2$) follows from two complementary congruences (preserved partitions), as in figure 3. Let $ab = c, ba = d$, then $\{a = c, b = d\}$ with image $L2$, and $\{a = d, b = c\}$ with image $R2$. Equivalently, the direct product is represented by two independent components $x = [x_1, x_2]$ the first composing as $L2$ and the second as $R2$.

The left- and right equivalences can be plotted pairwise in the plane as shown in fig 3, which also gives the composition tables of $L2, R2$ and $L2 \times R2 = \{a, b, ab, ba\}$. From this rectangular display follows the term **diagonal equivalence** for two indirectly equivalent invariants, since this is the only other form of equivalence. It is denoted by xDy where x and y are obtained by commutation: $x = ab$ and $y = ba$ for some a and b , which themselves are also diagonal equivalent aDb , with $a = aba = abba = xyandb = bab = baab = yx$. Diagonal equivalence occurs in pairs: if aDb then $abDba$ and vice versa.

(c) For $k > 2$ the above analysis can be generalized simply to $Lm \times Rn$ for $m.n$ invariants, with each invariant pair forming either $L2$ or $R2$ or $L2 \times R2$. Then $(ab)^k = ab$ and $(ba)^k = ba$ for some $k > 1$, where ab and ba are not nec. invariant, generating invariants $(ab)^{k-1} = ab^0$ and $(ba)^{k-1} = ba^0$ in a $k - 1$ cycle, with $(aba)^k = a$ and $(bab)^k = b$.

invariants $a \in S \setminus Z$ and $b \in Z$, then invariant aba is also in Z and has the same (minimal) rank as b , so $\text{rank}(a) > \text{rank}(aba) = \text{rank}(b)$. Hence strict ordering $a > aba$ holds. \square

The rectangle of equivalent pairs of invariants generalizes to $Lm \times Rn$, with $m, n \geq 2$. The $m \cdot n$ invariants form an $m \times n$ matrix, where L - $[R]$ equivalence holds between elements in the same column [row]. This is the general structure of a constant rank invariant semigroup (also called a rectangular 'band'):

THEOREM 3.1 *The following conditions on a finite semigroup S are equivalent:*

- (a) S is anti commutative (no two elements commute: $ab = ba$ implies $a = b$).
- (b) S is invariant and of constant rank.
- (c) $aba = a$ for all $a, b \in S$.
- (d) Each pair a, b of invariants in S is equivalent: either directly, forming $L2$ or $R2$, or indirectly (diagonal) via ab and ba forming $L2 \times R2$.
- (e) S is a direct product $Lm \times Rn$ of a left- and a right copy semigroup ($m, n \geq 1$).

Notice that rather general conditions (a)(b) imply a very regular structure (e), which is due to the strong properties of *finite* (rank) *associative* (semigroup) algebra.

PROOF. .

(a) \Rightarrow (b) : an anti- commutative semigroup S is invariant, because any iteration class x^+ is a commutative subsemigroup, so $|x^+|=1$ for all x , so each element of S is invariant. Moreover, S is of constant rank; otherwise some pair of invariants a, b would be properly ordered (lemma 4b) and thus commute, contradicting S being anti-commutative.

(b) \Rightarrow (c) : lemma 3.3b

(c) \Rightarrow (d) : $aba = a$ for all $a, b \rightarrow$ pairwise L -, R - or D - equivalent (lemma 3.3b).

(d) \Rightarrow (e) : Pairwise equivalence in S implies the direct product structure $Lm \times Rn$ with $m, n \geq 1$ as follows. If S contains only left- equivalent invariants then $S = Lm$ where $m = |S|$ and $n=1$. The other trivial case occurs when S contains n right equivalent invariants, and no left equivalence holds: $S = Rn$ with $m=1$ and $n = |S|$.

If both left- and right equivalences occur, the $Lm \times Rn$ rectangular structure (fig.3b) is seen as follows. Take any invariant z and form two subsets: Lz with all elements y that are left equivalent yLz to z , and Rz containing all x with xRz : right equivalent to z . They intersect only in z , because if w is left- and right equivalent to z , then it cannot differ from z : $w = wz = z.Lz$ and Rz are left- and right copy subsemigroups of S . Let the orders be respectively $|Lz| = m$ and $|Rz| = n$. Pairwise equivalence implies n copies of Lz which form a congruence λ of S with image $S/\lambda = Rn$. Similarly, congruence ρ consists of m copies of Rz , yielding image $S/\rho = Lm$. Since no pair of invariants can be both left- and right equivalent, congruences λ and ρ are orthogonal: $S = Lm \times Rn$.

(e) \Rightarrow (a) : semigroup $S = Lm \times Rn$ consists of pairwise equivalent invariants. Then it is anti- commutative which means that no pair commutes. For assume that one pair of distinct invariants a, b commutes: $ab = ba$, then they are either ordered $a < b$ or $a > b$ (in case ab is a or b), or their product is a third invariant $c = ab = ba$, their **meet**, that is ordered $c < a$ and $c < b$. Either case contradicts pairwise equivalence. \square

4 Maximal Subgroups: periodic G

LEMMA 4.1 of the iterations a^i of a semigroup element a with increasing i :

- the tail elements (if any) reduce strictly in rank, and
- the cycle elements (at least one: the invariant of a) have constant minimum rank.

PROOF. Consider the successive ranges Qa^i which, due to range lemma 1.1a, form a reducing inclusion chain of subsets of Q . Each range is contained properly in the previous one until the cycle is reached at $i = t+1$. Because as soon as two successive ranges are equal, then so are all next ranges: $Qa^i = Qa^{i+1} \longrightarrow Qa^{i+1} = Qa^{i+2}$, etc. (compose left and right by a). Once the cycle is reached, the minimum rank is obtained: the initial tail ranks decrease strictly, and all periodic elements in the cycle have equal and minimal rank. \square

COROLLARY 4.1 In a simple semigroup S every element is periodic (has no tail).

This follows directly from the previous lemma and lemma 1.1d, because if an element of S had a tail, then its iterations would have different ranks, which contradicts the constant rank property of a simple semigroup. To show that a simple semigroup is a disjoint union of isomorphic groups, we first need:

LEMMA 4.2 (maximal subgroups). Let S be a semigroup, then:

- (a) Periodic elements generating the same invariant e form a maximal subgroup of S , called the group G_e of e .
- (b) Equivalent invariants $a \sim b$ have isomorphic groups $G_a \cong G_b$:
 - if aLb via isomorphism $a.G_b = G_a$, mapping x in G_b to $ax \in G_a$;
 - if aRb via isomorphism $G_b.a = G_a$, mapping x in G_b to $xa \in G_a$;
 - if aDb via isomorphism $a.G_b.a = G_a$, mapping x in G_b to $axa \in G_a$.

PROOF. (a) Let periodic element x generate invariant e with period p , so $x^p = e$. Then clearly the inverse of x with respect to e is x^{p-1} . Define $x^0 = e$ for consistency in case $p = 1$ ($x = e$), and denote the inverse of x by x^{-1} . If y is another periodic element generating e , with inverse y^{-1} , then xy has inverse $(xy)^{-1} = y^{-1}.x^{-1}$ since $xy.(xy)^{-1} = x.y.y^{-1}.x^{-1} = x.e.x^{-1} = x.x^{-1} = e$, and similarly $(xy)^{-1}.xy = e$. It follows that xy generates the same invariant as x and y , so closure holds. Inverses are unique, because if x has two inverses x_1 and x_2 then $x_1 = x_1.e = x_1.(x.x_2) = (x_1.x).x_2 = e.x_2 = x_2$. So all periodic elements generating the same invariant form a group.

(b) Let a and b be two right equivalent invariants aRb so $ab = b$ and $ba = a$, then right composition of G_a with b is a morphism from G_a onto G_b , meaning G_b is an image of G_a , denoted $G_b|G_a$ (divisor relation). This follows, because a is identity for each y in $G_a : ay = ya = y$, while for each $x, y \in G_a : xb.yb = xb.a.yb = x.ba.yb = x.a.yb = xy.b$ (*), where $ba = a$ is used. In other words: the image of a composition of elements is the composition of their images.

We need $ab = b$ to show that xb is in G_b , in fact xb generates b upon iteration. This is seen by replacing y in (*) with x , then $(xb)^2 = (x^2).b$, and in general $(xb)^i = (x^i).b$. Let p be the period of x in $G_a : x^p = a$, then $(xb)^p = (x^p).b = ab = b$ in G_b .

So if $ab = b$ and $ba = a$ then a and b are right-copyers for each other, forming right equivalent invariants aRb , then right composition of G_a with b yields image G_b . Similarly, right composition of G_b with a yields image G_a . Consequently right equivalent invariants aRb have mutually ordered groups $G_b|G_a$ and $G_a|G_b$, so they are isomorphic: $G_a \cong G_b$.

Using left composition by a and b respectively, it follows that also left equivalent invariants have isomorphic groups. And finally, by transitivity, diagonal equivalent invariants have isomorphic groups as well. In that case aDb with (fig.3b) $aLba$, $baLb$ and $a.G_b.a = a.G_{ba} = G_a$. The diagonal case covers the other two cases of direct equivalence. \square

Conclusion: Combining all results, it follows that:

THEOREM 4.1 *The following conditions on a finite semigroup S are equivalent:*

- (a) S is simple (has no proper ideal).
- (b) S is of constant rank.
- (c) S is a disjoint union of isomorphic groups. Invariants (group-identities) form $L \times R$.
- (d) for invariants $a, b \in S$: $G_a = aSa$ and $aG_b.a = G_a$
- (e) S is a semi-direct product $(L \times R) \triangleright G$ of a left- and right-copy semigroup with a group.

PROOF. (a) \Rightarrow (b) : Corollary 2.1 (lemma 1.1).

(b) \Rightarrow (c) : Each element x of a constant rank semigroup S is periodic (cor 4.1). Hence S is a union of as many maximal subgroups as there are invariants, being the subgroup identities (lemma 4.2a). The subgroups are disjoint because no element can generate two invariants. Constant rank implies that no two invariants are ordered (cor 3.1), hence they are pairwise equivalent and form a direct product $L \times R$ (theorem 3.1e).

(c) \Rightarrow (d) : Consider an invariant a and elements of form $aSa = \{axa, x \in S\}$. Let the invariant generated by axa be $c = (axa)^p$ with period p . Since c begins and ends with invariant a , we have $ac = ca = c$, meaning $a \geq c$, and in fact $a = c$, since no strict ordering occurs in a constant rank semigroup. Hence $(axa)^p = a$, in other words axa generates invariant a for each x , and is thus in G_a . So for each x in constant rank semigroup S , axa is in the max-subgroup containing a , denoted as $aSa = G_a$.

If a, b are two equivalent invariants, with maximal subgroups G_a and G_b , then the group isomorphism is $aG_b.a = G_a$ with $axa = y$, independent of whether it is a left-, a right- or a diagonal equivalence (lemma 4.2b), the last case covers the first two.

(d) \Rightarrow (e) : Constant rank semigroup S contains as many disjoint isomorphic groups G as there are invariants, which are the group identities forming a direct product $L \times R$ subsemigroup (c). The direct product structure $L \times R \times G$ follows because there are two orthogonal congruences α and β with images $S/\alpha = L \times R$ and $S/\beta = G$.

(e) \Rightarrow (a) : Without going into detail here, in general the direct product of simple semigroups is also a simple semigroup [2, p83, ex8]. Since L, R and G are simple, so is their direct product. Moreover $L \times R$ can be seen to be an image but not necessarily a subsemigroup, with a coupling from $(L \times R)$ to G , hence a semi-direct product. \square

Any set A of state transforms that generate a constant rank closure, is a constant rank state machine $M(A, Q)$. As shown above, in general the closure $S = A^+/Q = (L \times R) \triangleright G$.

It is easily verified that Lm has m generators and $m+1$ states (see $L2$, fig.1) with the function of an m -branch; Rn has n generators and n states with an n -reset function, while group G has a permutation machine as generator with $k \leq |G|$ states. Then M is represented over $m + 1 + n + k$ states since L, R, G are 'relative prime' (have pairwise no common image, not proven here), and :

COROLLARY 4.2 *A general constant rank state machine M has a semi-direct product closure $(L \times R) \triangleright G$. It is the composition of machines with closures L, R, G respectively: a branch machine, a reset machine and a permutation machine.*

Further research

The decreasing-rank basic types of machines (fig.1): monotone iterative type U, and combinational logic type H (for instance embedding a lower semi-lattice in a boolean lattice), still need to be included, in order to obtain a general structure theory of State Machines. Of course, input and output logic functions should be taken into consideration as well [3] to yield an efficient overall logic design.

In essence, *associative algebra* and the *theory of finite semigroups* [2] need to be translated to *state machine language*, and applied to sequential logic synthesis, similar to the application of boolean algebra to the design of combinational logic circuits. This has been tried before, but with little practical impact, for the following reasons.

Using semigroup theory, Krohn-Rhodes [4] [5] derived, in the sixties, a prime decomposition theorem using only permutation and reset components, restricted further to cascade coupling. Clearly this is not a sufficient level of detail for practical purposes. All five basic component types [1] should be used, for a natural decomposition.

Also *loop coupling* of components may be required, for instance to decompose the '*prime*' *permutation machines* that are not cascade decomposable (with a 'simple group' closure). They can be very complex [1], and are not useful as practical network components.

References

1. N.Benschop: "On State Machine Decomposition and the Five Primitives of Sequential Logic", Int'l *Workshop on Logic Synthesis, MCNC*, USA, 1987.
2. A.Clifford, G.Preston: "*The Algebraic Theory of Semigroups*" Vol I, Mathematical Survey no.7 (AMS) 1961.
3. N.Benschop: "Min-Cut Algorithm for State Coding", Int'l *Workshop on Logic Synthesis, MCNC*, Research Triangle Park, NC, USA, 1989.
4. K.Krohn, J.Rhodes: "*Algebraic Theory of Machines*" part I, Tr.AMS Vol 116, pp 450-464 (1965).
5. A.Ginzburg: "*Algebraic Theory of Automata*", Academic Press, New York, 1968.